

Polarization operator of a photon in a magnetic field

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December 11, 2015

Abstract

In the first order of α , the polarization operator of a photon is investigated in a constant and homogeneous magnetic field at arbitrary photon energies. For weak and strong fields H (compared with the critical field $H_0 = 4.41 \cdot 10^{13}$ G), approximate expressions have been found. We consider the pure quantum region of photon energy near the threshold of pair creation, as well as the region of high energy levels where the quasiclassical approximation is valid. The general formula has been obtained for the effective mass of photon with given polarization. It is useful for an analysis of the problem under consideration on the whole and at a numerical work

1 Introduction

The study of QED processes in a strong magnetic field close to and exceeding the critical field strength $H_0 = m^2/e = 4.41 \cdot 10^{13}$ G (the system of units $\hbar = c = 1$ is used) is stimulated essentially by the existence of very strong magnetic field in nature. It is universally recognized the magnetic field of neutron stars (pulsars) run up $\sim 10^{11} \div 10^{13}$ G [1]. These values of field strength gives the rotating magnetic dipole model, in which the pulsar loses rotational energy through the magnetic dipole radiation. The prediction of this model is in quite good agreement with the observed radiation from pulsars in the radio frequency region. There are around some thousand radio pulsars. Another class of neutron stars, now referred to as magnetars [2], was discovered on examination of the observed radiation at x-ray and γ -ray energies and may possess even stronger surface magnetic fields $\sim 10^{14} \div 10^{15}$ G. The photon propagation in these fields and the dispersive properties of the space region with magnetic is of very much interest. This propagation accompanied by the photon conversion into a pair of charged particles when the transverse photon momentum is larger than the process threshold value $k_{\perp} > 2m$. When the field change is small on the characteristic length of process formation (for example, when this length is smaller then the scale of heterogeneity of the neutron star magnetic field), the consideration can be realized in the constant field approximation. In 1971 Adler [3] had calculated the photon polarization operator in a magnetic field using the proper-time

technique developed by Schwinger [4]. In the same year Batalin and Shabad [5] had calculated this operator in an electromagnetic field using the Green function found by Schwinger [4]. In 1975 the contribution of charged-particles loop in an electromagnetic field with n external photon lines had been calculated in [6]. For $n = 2$ the explicit expressions for the contribution of scalar and spinor particles to the polarization operator of photon were given in this work. For the contribution of spinor particles obtained expressions coincide with the result of [5], but another form is used.

The polarization operator in a constant magnetic field was investigated well enough in the energy region lower and near the pair creation threshold (see, for example, the papers [7, 8, 9] and the bibliography cited there. In the present paper we consider in detail the polarization operator on mass shell ($k^2 = 0$, the metric $ab = a^0b^0 - \mathbf{a}\mathbf{b}$ is used) at arbitrary value of the photon energy and magnetic field strength. The restriction of our consideration is only the applicability of the perturbation theory over the electromagnetic interaction constant α .

2 General expressions for the polarization operator

Our analysis is based on the general expression for the contribution of spinor particles to the polarization operator obtained in a diagonal form in [6] (see Eqs. (3.19), (3.33)). For the case of pure magnetic field we have in a covariant form the following expression

$$\Pi^{\mu\nu} = - \sum_{i=2,3} \kappa_i \beta_i^\mu \beta_i^\nu, \quad \beta_i \beta_j = -\delta_{ij}, \quad \beta_i k = 0; \quad (1)$$

$$\begin{aligned} \beta_2^\mu &= (F^* k)^\mu / \sqrt{-(F^* k)^2}, \quad \beta_3^\mu = (F k)^\mu / \sqrt{-(F k)^2}, \\ FF^* &= 0, \quad F^2 = F^{\mu\nu} F_{\mu\nu} = 2(H^2 - E^2) > 0, \end{aligned} \quad (2)$$

where $F^{\mu\nu}$ – the electromagnetic field tensor, $F^{*\mu\nu}$ – dual tensor, k^μ – the photon momentum, $(F k)^\mu = F^{\mu\nu} k_\nu$,

$$\kappa_i = \frac{\alpha}{\pi} m^2 r \int_{-1}^1 dv \int_0^{\infty - i0} f_i(v, x) \exp[i\psi(v, x)] dx. \quad (3)$$

Here

$$f_2(v, x) = 2 \frac{\cos(vx) - \cos x}{\sin^3 x} - \frac{\cos(vx)}{\sin x} + v \frac{\cos x \sin(vx)}{\sin^2 x},$$

$$f_3(v, x) = \frac{\cos(vx)}{\sin x} - v \frac{\cos x \sin(vx)}{\sin^2 x} - (1 - v^2) \cot x,$$

$$\psi(v, x) = \frac{1}{\mu} \left\{ 2r \frac{\cos x - \cos(vx)}{\sin x} + [r(1 - v^2) - 1]x \right\}; \quad (4)$$

$$r = -(F^* k)^2 / 2m^2 F^2, \quad \mu^2 = F^2 / 2H_0^2. \quad (5)$$

The real part of κ_i determines the refractive index n_i of the photon with polarization $e_i = \beta_i$:

$$n_i = 1 - \frac{\text{Re}\kappa_i}{2\omega^2}. \quad (6)$$

At $r > 1$ the proper value of polarization operator κ_i includes the imaginary part which determines the probability per unit length of pair production by photon with the polarization β_i :

$$W_i = -\frac{1}{\omega} \text{Im}\kappa_i \quad (7)$$

For $r < 1$ the integration counter over x in Eq. (3) may be turn to the lower axis ($x \rightarrow -ix$), then the value κ_i becomes real in the explicit form.

3 Weak field and low energy: $\mu \ll 1$, $1 < r \ll 1/\mu^2$

Let's remove the integration counter over x in Eq. (3) to the lower axis at the value x_0 :

$$x_0(r) = -il(r), \quad l(r) = \ln \frac{\sqrt{r} + 1}{\sqrt{r} - 1}. \quad (8)$$

As a result we have the following expression for κ_i :

$$\kappa_i = \frac{\alpha}{\pi} m^2 r (a_i + b_i), \quad (9)$$

where

$$a_i = -i \int_{-1}^1 dv \int_0^{l(r)} dx f_i(v, -ix) \exp[i\psi(v, -ix)], \quad (10)$$

$$b_i = \int_{-1}^1 dv \int_0^\infty dz f_i(v, z + x_0) \exp[i\psi(v, z + x_0)]. \quad (11)$$

In the integral a_i in Eq. (10) the small values $x \sim \mu$ contribute. This integral we calculate expanding the entering functions over x . Taking into account that in the region under consideration the condition $r\mu^2 \ll 1$ is fulfilled we keep in the exponent argument the term $-x/\mu$ only and extend the integration over x to infinity. In the result of not complicated integration over v we have:

$$\begin{aligned} a_2 &= -\frac{16}{45}\mu^2, & a_3 &= -\frac{28}{45}\mu^2; \\ \kappa_2^a &= -\frac{4\alpha m^2 \kappa^2}{45\pi}, & \kappa_3^a &= -\frac{7\alpha m^2 \kappa^2}{45\pi}, & \kappa^2 &= -\frac{(Fk)^2}{H_0^2 m^2}. \end{aligned} \quad (12)$$

In the integral b_i Eq. (11) the small values v contribute. Expanding entering functions over v and extending the integration over v to infinity we have

$$b_i = \sqrt{\mu\pi} \exp\left(-i\frac{\pi}{4}\right) \int_0^\infty \frac{dz f_i(0, x_0 + z)}{\sqrt{\chi(x_0 + z)}} \exp\left[-\frac{i}{\mu}\varphi(x_0 + z)\right], \quad (13)$$

$$\varphi(x) = 2r \tan\left(\frac{x}{2}\right) + (1-r)x, \quad \chi(x) = rx \left(1 - \frac{x}{\sin x}\right).$$

We consider now the energy region where $r - 1 \ll 1$ when the moving of created particles is nonrelativistic. In this case

$$\begin{aligned} i\varphi(z + x_0)/\mu &\simeq \beta(r) - \gamma(e^{-iz} + iz), \quad \chi(x) \simeq z + x_0; \\ \beta(r) &= 2\sqrt{r}/\mu + \gamma(1 - l(r)), \quad \gamma = (r - 1)/\mu, \\ f_2(0, x) &\simeq 0, \quad f_3(0, x) \simeq -i.. \end{aligned} \quad (14)$$

In the threshold region ($\gamma \sim 1$), where the particles occupy not very high energy levels, we present Eq. (13) for b_3 in the form

$$\begin{aligned} b_3 &= -i\sqrt{\mu\pi} \exp\left(-i\frac{\pi}{4} - \beta(r)\right) \\ &\times \int_0^\infty \frac{dz}{\sqrt{x_0 + z}} \sum_{n=0}^\infty \frac{\gamma^n}{n!} \exp[i(\gamma - n)z], \end{aligned} \quad (15)$$

The integral in Eq. (15) has a root singularity at whole numbers of $\gamma = k$. For $|\gamma - k| \ll x_0^{-1}$, $z \sim |\gamma - k|^{-1} \gg x_0$ we have:

$$\begin{aligned} b_3 &= -2i\sqrt{\mu\pi} \exp\left[-i\frac{\pi}{4} - \beta(r)\right] \frac{\gamma^k}{k!} \int_0^\infty dy \exp[i(\gamma - k)y^2] \\ &= -\pi \sqrt{\frac{\mu}{|\gamma - k|}} \exp\left[-\beta(r) + i\frac{\pi}{2}\vartheta(\gamma - k)\right] \frac{k^k}{k!}, \end{aligned} \quad (16)$$

where $\vartheta(z)$ — Heavside function: $\vartheta(z) = 1$ for $z \geq 0$, $\vartheta(z) = 0$ for $z < 0$. The expression for κ_3 with the accepted accuracy can be rewritten in the following form:

$$\begin{aligned} \kappa_3^b &\simeq -\alpha m^2 \mu e^{-\zeta} \sum_{n=0}^\infty \frac{(2\zeta)^n}{\sqrt{|g|} n!} \exp\left[i\frac{\pi}{2}\vartheta(g)\right], \\ g &= r - 1 - n\mu, \quad \zeta = 2r/\mu. \end{aligned} \quad (17)$$

At $\gamma \gg 1$ the small z contributes to the integral in Eq. (13), then:

$$\begin{aligned} i\varphi(z + x_0)/\mu &\simeq p(r) + \gamma z^2/2, \quad i\chi(z + x_0) \simeq ix_0 = l(r), \\ p(r) &= \frac{2\sqrt{r}}{\mu} - \gamma l(r), \quad \gamma = \frac{r - 1}{\mu}, \quad l(r) = \ln \frac{\sqrt{r} + 1}{\sqrt{r} - 1}. \end{aligned} \quad (18)$$

In the result of simple integration over z we have

$$\kappa_3^b \simeq -i\alpha m^2 e^{-p(r)} \sqrt{\frac{\mu}{2\gamma l(r)}}, \quad \kappa_3^b \simeq 0. \quad (19)$$

In a very wide range of energies, when the condition $1 \lesssim r-1 \ll \mu^{-2}$ is fulfilled, in Eq. (11) for b_i one can carry out the expansion over v and z from the very outset. As a result we have (see [10], Eq. (B5)):

$$\kappa_3^b \simeq -i\alpha m^2 e^{-p(r)} \sqrt{\frac{r}{\gamma l(r)p(r)}}, \quad \kappa_2^b = \frac{r-1}{2r} \kappa_3^b. \quad (20)$$

At $r-1 \ll 1$, the last equation for κ_i is consistent with the previous expression.

4 Weak field and high energy: $\mu \ll 1$, $r \gtrsim 1/\mu^2$

This region is contained in the region of the standard quasiclassical approximation (SQA) [10], [11]. The main contribution to the integral in Eq. (3) is given by small values of x . Expanding the entering functions Eq. (4) over x , and carrying out the change of variable $x = \mu t$, we get

$$\kappa_i = \frac{\alpha m^2 \kappa^2}{24\pi} \int_0^1 \alpha_i(v) (1-v^2) dv \int_0^\infty t \exp[-i(t + \xi \frac{t^3}{3})] dt; \quad (21)$$

$$\alpha_2 = 3 + v^2, \quad \alpha_3 = 2(3 - v^2),$$

$$\sqrt{\xi} = \frac{\kappa(1-v^2)}{4}, \quad \kappa^2 = 4r\mu^2 = -\frac{(Fk)^2}{m^2 H_0^2}. \quad (22)$$

The entering in Eq. (21) integrals over t are expressed by derivations of Airy (the imaginary part) and Hardy (the real part) integrals. Because of the application conditions, this energy region is overlapped with the considered above. At $\kappa \ll 1$ we have for the integrals entering into the real part of κ_i

$$\int_0^\infty t \cos t dt = -1, \quad \int_0^1 \alpha_2(1-v^2) dv = \frac{32}{15}, \quad \int_0^1 \alpha_3(1-v^2) dv = \frac{56}{15}. \quad (23)$$

These expression coincides with (12).

At calculation of the imaginary part of the integral over t in Eq. (21) we extend the integration to the whole axis because of the integrand parity. After that, the stationary phase method can be used ($t_0 = -i/\sqrt{\xi}$). As a result of the standard procedure of above method we have

$$\begin{aligned}
& \frac{1}{2} \int_{-\infty}^{\infty} t \exp[-i(t + \xi \frac{t^3}{3})] dt \\
&= \frac{t_0}{2} \sqrt{\frac{\pi}{it_0 \xi}} \exp[-i(t_0 + \xi \frac{t_0^3}{3})] \\
&= -i \frac{\sqrt{\pi}}{2} \xi^{-3/4} \exp\left(-\frac{2}{3\sqrt{\xi}}\right) = \\
&- 4i\sqrt{\pi}\kappa^{-3/2} (1-v^2)^{-3/2} \exp\left(-\frac{8}{3\kappa(1-v^2)}\right). \tag{24}
\end{aligned}$$

Substituting the obtained expression into Eq. (21) and fulfilling the integration over v , keeping in mind the small v contributes, we get:

$$\kappa_2 = -i\sqrt{\frac{3}{32}}\alpha m^2 \kappa \exp\left(-\frac{8}{3\kappa}\right), \quad \kappa_3 = 2\kappa_2. \tag{25}$$

At $\kappa \gg 1$ ($\xi \gg 1$) the small t contributes to the integral (21) ($\xi t^3 \sim 1$), and in the argument of exponent Eq. (21) the linear over t term may be omit. Carrying out the change of variable :

$$\xi t^3/3 = -ix, \quad t = \exp\left(\frac{-i\pi}{6}\right) \left(\frac{3x}{\xi}\right)^{1/3}, \tag{26}$$

one obtains:

$$\kappa_i = \frac{\alpha m^2 \kappa^2}{24\pi} \exp\left(\frac{-i\pi}{3}\right) \frac{1}{3} \left(\frac{48}{\kappa^2}\right)^{2/3} \Gamma\left(\frac{2}{3}\right) \int_0^1 dv \alpha_i(v) (1-v^2)^{-1/3}. \tag{27}$$

After integration over v we have:

$$\kappa_i = \frac{\alpha m^2 (3\kappa)^{2/3}}{7\pi} \frac{\Gamma^3\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} (1 - i\sqrt{3})\beta_i, \quad \beta_2 = 2, \quad \beta_3 = 3. \tag{28}$$

5 Strong fields: $\mu \gtrsim 1$

Let's consider the energy region the upper boundary of which is slightly higher the characteristic energy r_{10} . At this energy one of the particles is created on the first excited level and another particle is created on the lower level. At that we choose the lower boundary of the energy region slightly below the threshold energy r_{00} :

$$\begin{aligned}
r_{lk} &= (\varepsilon(l) + \varepsilon(k))^2/4m^2, \\
\varepsilon(l) &= \sqrt{m^2 + 2eHl} = m\sqrt{1 + 2\mu l}. \tag{29}
\end{aligned}$$

For $r < r_{10}$, the integration counter over x in Eq. (3) may be turn to the lower imaginary axis for all terms in the integrand except the term in expression for κ_3 , which contains

the function $\Phi(v, x) = -(1 - v^2)\text{ctg}x \exp[i\psi(v, x)]$. Let's add to $\Phi(v, x)$ and take off the function

$$\begin{aligned}\Phi_{\text{red}}(v, x) &= i(1 - v^2) \exp[i\psi_{\text{red}}(v, x)], \\ \psi_{\text{red}}(v, x) &= \frac{1}{\mu} \{2ri + [r(1 - v^2) - 1]x\}.\end{aligned}\quad (30)$$

After that in the integral over x for the sum $\Phi(v, x) + \Phi_{\text{red}}(v, x)$, the integration counter over x can be turn to the lower axis. The integral over x for the residuary function has the the following form

$$\begin{aligned}& \int_0^\infty \exp[i\psi_{\text{red}}(v, x)] dx \\ &= \exp\left(-\frac{2r}{\mu}\right) \frac{i\mu}{r(1 - v^2) - 1 + i0} \\ &= \mu \exp\left(-\frac{2r}{\mu}\right) \left[i \frac{\mathcal{P}}{r - 1 - rv^2} + \pi\delta(r - 1 - rv^2) \right].\end{aligned}\quad (31)$$

The operator \mathcal{P} means the principal value integral. Carrying out the integration over v , we have

$$\begin{aligned}& -ir \int_{-1}^1 dv (1 - v^2) \left[i \frac{\mathcal{P}}{r - 1 - rv^2} + \pi\delta(r - 1 - rv^2) \right] \\ &= 2 \left[1 + \frac{1}{\sqrt{r(1 - r)}} \arctan \sqrt{\frac{1 - r}{r}} \right] - \frac{\pi}{\sqrt{r(1 - r)}}.\end{aligned}\quad (32)$$

Finally the expression for κ_i takes the following well-behaved form:

$$\kappa_2 = \alpha m^2 \frac{r}{\pi_{-1}} \int_0^\infty dv \int_0^\infty F_2(v, x) \exp[-\chi(v, x)] dx, \quad \kappa_3 = \kappa_3^1 + \kappa_3^{00}, \quad (33)$$

$$\begin{aligned}\kappa_3^1 &= \alpha m^2 \frac{r}{\pi_{-1}} \int_0^\infty dv \int_0^\infty \{ F_3(v, x) \exp[-\chi(v, x)] \\ &+ (1 - v^2) \exp[-\chi_{00}(v, x)] dx \},\end{aligned}\quad (34)$$

$$\kappa_3^{00} = \alpha m^2 \frac{\mu}{\pi} \exp\left(-\frac{2r}{\mu}\right) [2 + B(r)]; \quad (35)$$

$$B(r) = \frac{2}{\sqrt{r(1 - r)}} \arctan \sqrt{\frac{1 - r}{r}} - \frac{\pi}{\sqrt{r(1 - r)}}. \quad (36)$$

At the photon energy higher threshold ($r > 1$, $\sqrt{1 - r} = -i\sqrt{r - 1}$):

$$B(r) = \frac{2}{\sqrt{r(r - 1)}} \ln(\sqrt{r} + \sqrt{r - 1}) - \frac{i\pi}{\sqrt{r(r - 1)}}. \quad (37)$$

Here

$$F_2(v, x) = \frac{1}{\sinh x} \left(2 \frac{\cosh x - \cosh(vx)}{\sinh^2 x} - \cosh(vx) + v \sinh(vx) \coth x \right),$$

$$F_3(v, x) = \frac{\cosh(vx)}{\sinh x} - v \frac{\cosh x \sinh(vx)}{\sinh^2 x} - (1 - v^2) \coth x; \quad (38)$$

$$\chi(v, x) = \frac{1}{\mu} \left[2r \frac{\cosh x - \cosh(vx)}{\sinh x} + (rv^2 - r + 1)x \right], \quad (39)$$

$$\chi_{00}(v, x) = \frac{1}{\mu} [2r + (rv^2 - r + 1)x]. \quad (40)$$

For superstrong fields ($\mu \gg 1$), the entering into integrands of Eqs. (33) and (34) exponential terms can be substitute for unit. As a result we have for leading terms

$$\kappa_2 \simeq -\frac{4r}{3\pi} \alpha m^2, \quad \kappa_3 \simeq -\alpha m^2 \frac{\mu}{\pi} (2 + B(r)). \quad (41)$$

The integrals for κ_2 and κ_3^1 have the root divergence at $r = r_{10}$. To bring out these distinctions in an explicit form, let's consider the main asymptotic terms of corresponding integrand at $x \rightarrow \infty$:

$$\kappa_i^{10} = \alpha m^2 r \frac{2}{\pi} \int_{-1}^1 dv \int_0^\infty d_i(v) \exp[-\chi_{10}(v, x)] dx, \quad (42)$$

$$d_2 = v - 1, \quad d_3 = 1 - v - \frac{2r}{\mu} (1 - v^2) \quad (43)$$

$$\chi_{10}(v, x) = \frac{2r}{\mu} + \frac{1}{\mu} [(1 - v)\mu + rv^2 - r + 1] x. \quad (44)$$

After elementary integration over x , one gets

$$\kappa_i^{10} = \alpha m^2 \mu r \frac{2}{\pi} \exp\left(-\frac{2r}{\mu}\right) \int_{-1}^1 dv \frac{d_i(v)}{rv^2 - \mu v - r + 1 + \mu}. \quad (45)$$

Performing integration over v , we have:

$$\kappa_2^{10} = \alpha m^2 \mu r \frac{2}{\pi} \exp\left(-\frac{2r}{\mu}\right) \left[\frac{\mu/2r - 1}{\sqrt{h(r)}} A(r) - \frac{1}{2r} \ln(2\mu + 1) \right], \quad (46)$$

$$\begin{aligned} \kappa_3^{10} &= \alpha m^2 \mu r \frac{2}{\pi} \exp\left(-\frac{2r}{\mu}\right) \\ &\times \left[\frac{\mu/2r - 1 - 2/\mu}{\sqrt{h(r)}} A(r) - \frac{1}{2r} \ln(2\mu + 1) + \frac{2}{\mu} \right], \end{aligned} \quad (47)$$

$$\begin{aligned} A(r) &= \arctan \frac{r - \mu/2}{\sqrt{h(r)}} + \arctan \frac{r + \mu/2}{\sqrt{h(r)}} \\ &= \pi - \arctan \frac{\sqrt{h(r)}}{r - \mu/2} - \arctan \frac{\sqrt{h(r)}}{r + \mu/2}, \end{aligned} \quad (48)$$

$$h(r) = (1 + \mu)r - r^2 - \mu^2/4. \quad (49)$$

At $r = r_{10} = (1 + \mu + \sqrt{1 + 2\mu})/2$, $h(r) = 0$ and expressions in Eqs. (46)-(47) contain the root divergence :

$$\kappa_i^{10} \simeq -4\alpha m^2 r \exp\left(-\frac{2r}{\mu}\right) \frac{\beta_i}{\sqrt{h(r)}}, \quad \beta_2 = \frac{\mu}{2} - \frac{\mu^2}{4r}, \quad \beta_3 = 1 + \frac{\mu}{2} - \frac{\mu^2}{4r}. \quad (50)$$

For the higher photon energy $r > r_{10}$ (but $r < r_{20} = (1 + \sqrt{1 + 4\mu})^2/4$), the new channel of pair creation arises, and Eq. (45) changes over (cf. (32)):

$$\begin{aligned} \kappa_i^{10} &= \alpha m^2 \mu r \frac{2}{\pi} \exp\left(-\frac{2r}{\mu}\right) \\ &\times \int_{-1}^1 dv d_i(v) \left[\frac{\mathcal{P}}{rv^2 - \mu v - r + 1 + \mu} \right. \\ &\left. - i\pi \delta(rv^2 - \mu v - r + 1 + \mu) \right]; \end{aligned} \quad (51)$$

At $r - r_{10} \ll 1$

$$\kappa_i^{10} \simeq -4i\alpha m^2 r \exp\left(-\frac{2r}{\mu}\right) \frac{\beta_i}{\sqrt{-h(r)}}. \quad (52)$$

This direct procedure of divergence elimination can be extended further. But we consider, in the next section, another technique allowing to perform done extracting in general case.

For strong fields and high energy levels ($\mu \gtrsim 1$, $r \gg \mu$), Eqs. (26-28) can be used because of $x \sim (\mu/r)^{1/3} \ll 1$ contributes, and condition $\kappa \gg 1$ is identically valid in this case. Formula (28) coincides with the corresponding formula of SQA at $\kappa \gg 1$. However, be aware that for weak fields ($\mu \ll 1$, $H \ll H_0$), condition $\kappa \gg 1$ is sufficient for the quasiclassical motion ($n \gg 1$) of produced particles. While for the fields significantly larger than the critical field ($\mu \gg 1$), a large value of the parameter κ does not provide specified quasiclassicality. In this case, a prerequisite for the applicability of the SQA is the condition $r/\mu \sim n \gg 1$.

6 General case

As well as in our work [10] (see Appendix A), we present the effective mass in the form of

$$\kappa_i = \alpha m^2 \frac{r}{\pi} T_i; \quad T_i = \int_{-1}^1 dv \int_0^{\infty - i0} f_i(v, x) \exp[i\psi(v, x)] dx, \quad (53)$$

$$T_i = \sum_{n=0}^{\infty} \left(1 - \frac{\delta_{n0}}{2}\right) T_i^{(n)}; \quad (54)$$

$$T_i^{(n)} = \int_{-1}^1 dv \int_0^{\infty - i0} F_i^{(n)}(v, x) \exp[ia_n(v)x] dx, \quad (55)$$

where

$$\begin{aligned} F_1^{(n)} &= (-i)^n \exp(iz \cot x) \left[\frac{i}{\sin x} (J_{n+1}(t) - J_{n-1}(t)) - \frac{2vn}{z} \cot x J_n(t) \right], \\ F_2^{(n)} &= (-i)^n \exp(iz \cot x) \frac{4}{z} \left(b \cot x - \frac{i}{\sin^2 x} \right) J_n(t) - F_1^{(n)}, \\ F_3^{(n)} &= F_1^{(n)} - 2(-i)^n \exp(iz \cot x) (1 - v^2) \cot x J_n(t); \end{aligned} \quad (56)$$

$$a_n(v) = nv - b, \quad b = \frac{1}{\mu} (1 - r(1 - v^2)), \quad z = \frac{2r}{\mu}, \quad t = \frac{z}{\sin x}. \quad (57)$$

Let's note that at $x \rightarrow -i\infty$ the asymptotic of the Bessel function $J_n(t)$ is

$$J_n(t) \simeq J_n(2ize^{-|x|}) \simeq \frac{(iz)^n}{n!} e^{-n|x|}, \quad (58)$$

and under the condition $a_n(v) < n$, the integration counter over x in Eq. (54) can be unrolled to the lower axis. Then $T_i^{(n)}$ becomes real in the explicit form.

The functions $F_i^{(n)}(v, x)$ are periodical over x . So one can present $T_i^{(n)}$ as

$$\begin{aligned} T_i^{(n)} &= \int_{-1}^1 dv \int_0^{2\pi} F_i^{(n)}(v, x) \exp[ia_n(v)x] dx \sum_{k=0}^{\infty} \exp[2\pi i k a_n(v)] \\ &= \int_{-1}^1 \frac{dv}{1 - \exp[2\pi i a_n(v)] + i0} \int_0^{2\pi} F_i^{(n)}(v, x) \exp[ia_n(v)x] dx. \end{aligned} \quad (59)$$

We use the well-known expression

$$\begin{aligned} &\frac{1}{1 - \exp[2\pi i a_n(v)] + i0} \\ &= \frac{\mathcal{P}}{1 - \exp[2\pi i a_n(v)]} - i\pi \delta(1 - \exp[2\pi i a_n(v)]), \end{aligned} \quad (60)$$

Taking into account the above notation (58), we have

$$\begin{aligned} & -i\pi\delta(1 - \exp[2\pi i a_n(v)]) \\ & = -i\pi \sum_m \delta(1 - \exp[2\pi i(a_n(v) - m)]) \end{aligned} \quad (61)$$

$$\rightarrow \frac{1}{2} \sum_{m \geq n} \delta(a_n(v) - m). \quad (62)$$

Also using the ratio $F_i^{(n)}(v, x + \pi) = (-1)^n F_i^{(n)}(v, x)$, we get

$$\begin{aligned} T_i^{(n)} &= (-1)^n \frac{i}{2} \mathcal{P} \int_{-1}^1 \frac{dv}{\sin(\pi a_n(v))} \int_{-\pi}^{\pi} F_i^{(n)}(v, x) \exp[ia_n(v)x] dx \\ &+ \sum_{m \geq n}^{m=n_{\max}} \sum_{v_{1,2}} \frac{1 + (-1)^{m+n}}{2|a'_n(v)|} \vartheta(g(n, m, r)) \end{aligned} \quad (63)$$

$$\times \int_{-\pi}^{\pi} F_i^{(n)}(v_{1,2}, x) \exp[imx] dx, \quad (64)$$

where

$$g(n, m, r) = r^2 - (1 + m\mu)r + n^2\mu^2/4, \quad (65)$$

$$v_{1,2} = \frac{n\mu}{2r} \pm \frac{1}{r} \sqrt{g}, \quad a'_n(v) = \frac{2}{\mu} \sqrt{g}; \quad (66)$$

$$n_{\max} = [d(r)], \quad d(r) = \frac{2(r - \sqrt{r})}{\mu}. \quad (67)$$

Here $[d]$ is the integer part of d .

Bringing out the distinction in the explicit form, we present $T_i^{(n)}$ as

$$T_i^{(n)} = T_i^{(nr)} + T_i^{(ns)}; \quad (68)$$

$$\begin{aligned} T_i^{(nr)} &= (-1)^n \frac{i}{2} \mathcal{P} \int_{-1}^1 dv \\ &\times \int_{-\pi}^{\pi} \left[F_i^{(n)}(v, x) \frac{\exp[ia_n(v)x]}{\sin(\pi a_n(v))} - \sum_{m \geq n}^{m=n_{\max}} \sum_{v_{1,2}} \frac{(-1)^m}{\pi} F_i^{(n)}(v_{1,2}, x) \frac{\exp[imx]}{a_n(v) - m} \right] dx, \end{aligned} \quad (69)$$

$$T_i^{(ns)} = \sum_{m \geq n}^{m=n_{\max}} \sum_{v_{1,2}} \frac{\mu\pi}{2\sqrt{g}} \left[1 - \frac{1}{\pi} \left(\arctan \frac{2\sqrt{-g}}{2r - \mu n} + \arctan \frac{2\sqrt{-g}}{2r + \mu n} \right) \right] \quad (70)$$

$$\times \int_0^{\pi} F_i^{(n)}(v_{1,2}, x) \exp[imx] dx. \quad (71)$$

Here the regularized function $T_i^{(nr)}$ is singularity-free, and for $n > n_{\max}$ the integration counter in $T_i^{(n)}$ can be unrolled to the lower axis. . After that we present T_i in the form

$$\begin{aligned}
T_i &= \sum_{n > n_{\max}}^{\infty} T_i^{(n)} + \sum_{n=0}^{n_{\max}} T_i^{(n)} = \left(T_i - \sum_{n=0}^{n_{\max}} T_i^{(n)} \right) + \sum_{n=0}^{n_{\max}} T_i^{(n)} \\
&= \int_{-1}^1 dv \int_0^{\infty} \left\{ F_i(v, x) \exp[-\chi(v, x)] + i \sum_{n=0}^{n_{\max}} F_i^{(n)}(v, -ix) \exp[a_n(v)x] \right\} dx \\
&\quad + \sum_{n=0}^{n_{\max}} T_i^{(n)}.
\end{aligned} \tag{72}$$

Here the functions $F_i(v, x)$, $\chi(v, x)$ are given by Eqs. (38), (39), and $a_n(v)$ by Eq. (57). The integrals over x in the expression for $T_i^{(ns)}$ have been calculated in Appendix A [10]. Along with integers m and n , we use also $l = (m + n)/2$ and $k = (m - n)/2$ which are straight the level numbers (see Eq. (29)). We have

$$\begin{aligned}
\kappa_i^s &= \alpha m^2 \frac{r}{\pi} \sum_{n=0}^{n_{\max}} \left(1 - \frac{\delta_{n0}}{2} \right) T_i^{(ns)} = -i\alpha m^2 \mu e^{-\zeta} \sum_{n,m} (2 - \delta_{n0}) \frac{\zeta^n k!}{\sqrt{g} l!} \\
&\quad \times \left[1 - \frac{1}{\pi} \left(\arctan \frac{2\sqrt{-g}}{2r - \mu n} + \arctan \frac{2\sqrt{-g}}{2r + \mu n} \right) \right] D_i;
\end{aligned} \tag{73}$$

$$\begin{aligned}
D_2 &= \left(\frac{m\mu}{2} - \frac{n^2\mu^2}{4r} \right) F \\
&\quad + 2\mu l \vartheta(k-1) [2L_{k-1}^{n+1}(\zeta) L_k^{n-1}(\zeta) - L_k^n(\zeta) L_{k-1}^n(\zeta)],
\end{aligned} \tag{74}$$

$$\begin{aligned}
D_3 &= \left(1 + \frac{m\mu}{2} - \frac{n^2\mu^2}{4r} \right) F + 2\mu l \vartheta(k-1) L_k^n(\zeta) L_{k-1}^n(\zeta), \\
F &= [L_k^n(\zeta)]^2 + \vartheta(k-1) \frac{l}{k} [L_{k-1}^n(\zeta)]^2, \quad \zeta = \frac{2r}{\mu},
\end{aligned} \tag{75}$$

where $L_k^n(\zeta)$ is the generalized Laguerre polynomial.

At $\mu \ll 1$, $(r-1)/\mu \lesssim 1$, $g/\mu \simeq |(r-1)/\mu - m| \ll 1$ the main terms of sum in Eq. (73) have a form:

$$\kappa_3^s \simeq -i\alpha m^2 \mu e^{-\zeta} \zeta^m g^{-1/2} \sum_{k+l=m} \frac{1}{k! l!} \tag{76}$$

$$= -i\alpha m^2 \mu e^{-\zeta} \zeta^m g^{-1/2} \frac{2^m}{m!}, \quad \kappa_2^s \simeq \frac{1}{2} m \mu \kappa_3^s. \tag{77}$$

Here we take into account that for $\zeta \gg 1$

$$L_k^n(\zeta) \simeq \zeta^k / k!, \quad D_3 \simeq [L_k^n(\zeta)]^2, \quad D_2 \simeq m \mu D_3 / 2. \tag{78}$$

Eq. (76) coincides with Eq. (17). Note that for $g > 0$, Eq. (76) (as the general Eq. (73)) gives in addition the partial probability of level population by created particles (see [10]).

At $\mu \gtrsim 1$, $|r - 1| \ll 1$, $m = n = k = l = 0$, $D_3 = 1$, $D_2 = 0$, and Eq. (73) coincides with Eq. (35). At $|r - r_{10}| \ll 1$, the main term of sum is $m = n = l = 1$, $k = 0$, $D_2 = \beta_2$, $D_3 = \beta_3$, $g = -h$, and this equation coincides with Eq. (52).

7 Conclusion

So, we have investigated the photon polarization operator in weak and strong magnetic fields for arbitrary values of the photon energy. At large quantum numbers in a weak ($H \ll H_0$, $\mu \ll 1$) field, there are two regions of the photon energy. In each of these, an approximate description is of different nature and thus a different form. The first area (with not very large quantum numbers) is adjacent to the region of the threshold energy. From this side it is the non-relativistic region ($r - 1 \ll 1$). The first area applicability ends on another side at relativistic energies ($r \gg 1$), when the parameter $\kappa = 2\mu\sqrt{r}$ is not small. For these energies, SQA is applicable, such that the energy regions of these approximations are intersect at $\kappa \ll 1$. In a weak field at $\kappa \sim 1$, the imaginary part of the polarization operator is expressed in terms of the derivative of the Airy function, the real part is related to the Hardy function. When $\kappa \gg 1$, the approximate description of the polarization operator is greatly simplified.

In strong fields in the expressions for the effective photon mass κ_i , we have identified integrals asymptotically diverging at the threshold energies and taking analytically. In the remaining integrals, the contour of integration can be moved on the imaginary axis, so that they become real explicitly. These integrals converge well as in the integrand instead of oscillating functions, we have exponentially falling functions. It is necessary for the analysis of received expressions and numerical calculations. With increasing of the photon energy, the procedure (used to the lower threshold r_{00} and r_{10}) could be extended in the next area to the higher thresholds of pair creation. However, a more consistent was the creation of the regular method to carry out the corresponding calculations in general form. The imaginary part of the polarization operator, obtained in a manner, coincides with the general formula for the probability of a photon pair [10].

This work was supported in part by the Ministry of Education and Science of the Russian Federation. The author is grateful to the Russian Foundation for Basic Research grant №15-02-02674 for partial support of the research.

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